A one-dimensional many-body integrable model from Z_n Belavin model with open boundary conditions

Heng Fan[a,b], **Bo-Yu Hou**[b], **Guang-Liang Li** [b],
Kang-Jie Shi[a,b], **Yan-Shen Wang**[a,c]
[a] CCAST(World Laboratory),
P. O. Box 8730, Beijing 100080, P. R. China
[b] Institute of Morden Physics, Northwest University
P. O. Box 105, Xi'an, 710069, P. R. China
[c] Zhejiang Institute of Morden Physics, Zhejiang University
Hangzhou, 310027, P. R. China

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Abstract

We use factorized L operator to construct an integrable model with open boundary conditions. By taking trigonometic limit($\tau \to \sqrt{-1}\infty$) and scaling limit($\omega \to 0$), we get a Hamiltonian of a classical integrable system. It shows that this integrable system is similar to those found by Calogero et al.

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I Introdction

In the last decades, a series of one-dimensional integrable many-body systems have been found [1-6]. One effective method to prove the integrability of the many-body systems is the Lax representation, which means that we can construct the complete set of integrals of the motion. The Lax representation for the elliptic Calogero-Moser model was found by Krichever [7]. And the Ruijsenaars-Macdonald's commuting difference operators can also prove the integrability of the many-body systems[5,6,8,9].

It is known that there are a lot of two-dimensional exactly sovled models in statistical mechanics[10-15]. The integrability is proved by the commuting transfer matrix. Recently, some relations between the one-dimensional many-body systems and the two-dimensional sovable models are found, see references [16-19] and the references therein. Hasegawa [16] found that the L operator [20-22] for the two-dimensional Z_n Belavin model can be related to the Krichever's Lax matrix [7]. And the commuting difference operators given by the L operators is similar to the Ruijsenaars-Macdonald's difference operators. Hasegawa studied the Z_n Belavin model by imposing the periodic boundary conditions. We will study the Z_n Belavin model by imposing the open boundary conditions, for the case of open boundary conditions see references [23,24] and the references therein. Using the factorized L operators and one solution of the reflection equation, we can constrcut a cummuting difference operators which should be equivalent to the Ruijsenaars-Macdonald's difference operators. By taking a special limit, we find a trigonometic integrable model which is similar to the trigonometric model found by Calogero et al [2-4]. Principally, we can obtain a classical integrable system if we can find a solution to the reflection equation.

The paper is organized as follows: In section 2, we will introduce the Z_n symmetric Belavin model. The factorized L operator will be given in section 3. The commuting difference operators connected with the transfer matrix with open boundary conditions will be given in section 4. The special limit is taken in section 5, the integrable model is found in this section. Section 6 containes a summary and discussions.

II The Z_n symmetric Belavin R matrix

The Z_n symmetric Belavin R matrix [14,15] is given as

$$R_{12} = \frac{1}{n} \sum_{\alpha \in Z_n^2} W_\alpha(z) I_\alpha \otimes I_\alpha^{-1}, \tag{1}$$

with

$$W_\alpha(z) = \frac{\sigma_\alpha(z+\eta)}{\sigma_\alpha(\eta)}, I_\alpha = g^{\alpha_2} h^{\alpha_1}, h_{ij} = \delta_{i+1,j},$$

$$g_{ij} = \omega^i \delta_{i,j}, \omega = e^{\frac{2\pi\sqrt{-1}}{n}}, (i, j \in Z_n),$$

$$\sigma_\alpha(z) \equiv \theta \left[\begin{array}{c} \frac{1}{2} + \frac{\alpha_1}{n} \\ \frac{1}{2} + \frac{\alpha_2}{n} \end{array} \right] (z, \tau),$$

$$\theta \left[\begin{array}{c} a \\ b \end{array} \right] (z, \tau) \equiv \sum_{m \in Z} e^{\pi\sqrt{-1}(m+a)^2\tau + 2\pi\sqrt{-1}(m+a)(z+b)}. \tag{2}$$

This R-matrix satisfy the Yang-Baxter equation(YBE) [10,11],

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2), \tag{3}$$

where $R_{12}(z), R_{13}(z)$ and $R_{23}(z)$ act in $C^n \otimes C^n \otimes C^n$ with $R_{12}(z) = R(z) \otimes 1, R_{23}(z) = 1 \otimes R(z)$, ect. One can find that the R-matrix also satisfy the following unitary and

cross-unitary properties.

$$R_{12}(z_1 - z_2)R_{21}(z_2 - z_1) = \rho(z_1 - z_2) \cdot id. \tag{4}$$

$$R_{21}^{t_2}(z_2 - z_1 - nw)R_{12}^{t_2}(z_1 - z_2) = \tilde{\rho}(z_1 - z_2) \cdot id. \tag{5}$$

where

$$\rho(z) = \frac{h(z + w)h(-z + w)}{h^2(w)} \tag{6}$$

$$\tilde{\rho}(z) = \frac{h(z)h(-z - nw)}{h^2(w)}, \tag{7}$$

w is defined by $w \equiv n\eta$ and $h(z) \equiv \sigma_0(z)$, t_i means the transposition in the i-th space.

Assume an operator matrix L(z) satisfy the Yang-Baxter relation(YBR)

$$R_{12}(z_1 - z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R_{12}(z_1 - z_2), \tag{8}$$

with $L_1(z_1) = L(z_1) \otimes 1, L_2(z_2) = 1 \otimes L(Z_2)$. For the periodic boundary conditions, its transfer matrix is defined by $t(z) = trL(z)$. We can prove that this tranfer matrices with different spectrum commute with each other $[t(z_1), t(z_2)] = 0$. For the open boundary conditions, Sklyanin [23] proposed a systematic approach to handle the problems which involves the reflection equation (RE)

$$R_{12}(z_1 - z_2)K_1(z_1)R_{21}(z_1 + z_2)K_2(z_2) = K_2(z_2)R_{12}(z_1 + z_2)K_1(z_1)R_{21}(z_2 - z_1), \tag{9}$$

where the reflecting K matrix is a solution of the RE. In order to construct the integrable models, we also need a dual reflection equation(DRE) which is associated with the cross-unitary relation of the R matrix. For Z_n belavin model, the dual RE takes the form [24],

$$R_{12}(z_1 - z_2)\tilde{K}_1(z_1)R_{21}(-z_1 - z_2 - nw)\tilde{K}_2(z_2)$$

$$= \tilde{K}_2(z_2)R_{12}(-z_1-z_2-nw)\tilde{K}_1(z_1)R_{21}(z_2-z_1). \quad (10)$$

If we define the transfer matrix as $t(z) = tr[\tilde{K}(z)L(z)K(z)L^{-1}(-z)]$, with the help of unitary and cross-unitary relations, one can prove that $[t(z_1), t(z_2)] = 0$, that means the model under consideration is integrable.

Additionally, there is an isomorphism between $K(z)$ and $\tilde{K}(z)$

$$\Phi : K(z) \to \tilde{K}(z) = K(-z - \frac{nw}{2}). \quad (11)$$

Given a solution K(z) of the RE, we can find a solution $\tilde{K}(z)$ of the DRE. For Z_n Belavin model, a solution K(z) to the RE is as follows[24],

$$K(z) = K_0(z)K_0(0), \quad (12)$$

where

$$K_0(z) = \sum_{\alpha\in Z_n^2} U_{2\alpha}(z)\omega^{2\alpha_1\alpha_2}I_{2\alpha}, \quad (13)$$

$$U_{2\alpha}(z) = \frac{\sigma_{2\alpha}(z+c)}{\sigma_{2\alpha}(c)}, \quad (14)$$

c is a arbitrary constant.

III The factorized L matrix

Jimbo et al [25] defined the intertwiner of Z_n model as an n-element column vector $\phi_{a,a+\hat{\mu}}(z)$ whose j-th element is

$$\phi^{(j)}_{a,a+\hat{\mu}}(z) = \theta^{(j)}(z - nw\bar{a}_\mu, n\tau), \quad (15)$$

$$\theta^{(j)}(z - nw\bar{a}_\mu, n\tau) = \theta\begin{bmatrix} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{bmatrix}(z - nw\bar{a}_\mu, n\tau), \quad (16)$$

$\bar{a}_\mu = a_\mu - \frac{1}{n}\sum_\nu a_\nu + \delta_\mu, a \in Z^n$, δ_μ's are some generic numbers. Using intertiwner, the face-vertex correspondence can be written as

$$\begin{aligned} & R_{12}(z_1 - z_2)_{ij}^{i'j'} \phi^{(i')}_{a-\hat{\mu}-\hat{\nu},a-\hat{\mu}}(z_1)\phi^{(j')}_{a-\hat{\mu},a}(z_2) \\ = & \sum_\kappa W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\nu} & a-\hat{\mu} \\ a-\hat{\kappa} & a \end{array}\right](z_1-z_2)\phi^{(j)}_{a-\hat{\mu}-\hat{\nu},a-\hat{\kappa}}(z_2)\phi^{(i)}_{a-\hat{\kappa},a}(z_1), \end{aligned} \tag{17}$$

where $W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\nu} & a-\hat{\mu} \\ a-\hat{\kappa} & a \end{array}\right](z)$ is the face Boltzmann weight of $A^{(1)}_{n-1}$ IRF model [25]. It is defined as

$$W\left[\begin{array}{cc} a-2\hat{\mu} & a-\hat{\mu} \\ a-\hat{\mu} & a \end{array}\right](z) = \frac{h(z+w)}{h(w)},$$

$$W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\nu} & a-\hat{\mu} \\ a-\hat{\mu} & a \end{array}\right](z) = \frac{h(z-a_{\mu\nu}w)}{h(-a_{\mu\nu}w)}, \tag{18}$$

$$W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\nu} & a-\hat{\nu} \\ a-\hat{\mu} & a \end{array}\right](z) = \frac{h(z)h((a_{\mu\nu}+1)w)}{h(w)h(a_{\mu\nu}w)}. \tag{19}$$

The other face Boltzmann weights are defined as zeroes. Where $a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu$. At the same time, we can also find the n-element row vectors $\tilde{\phi}$ and $\bar{\phi}$ which satisfy the following relations[20-22,25]

$$\tilde{\phi}^{(k)}_{a-\hat{\mu},a}(z)\phi^{(k)}_{a-\hat{\nu},a}(z) = \delta_{\mu\nu}, \tag{20}$$

$$\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(z)\phi^{(k)}_{a,a+\hat{\nu}}(z) = \delta_{\mu\nu}. \tag{21}$$

The above equation can also be written as

$$\sum_\mu \phi_{a-\hat{\mu},a}(z)\tilde{\phi}_{a-\hat{\mu},a}(z) = I, \tag{22}$$

$$\sum_\mu \phi_{a,a+\hat{\mu}}(z)\bar{\phi}_{a,a+\hat{\mu}}(z) = I. \tag{23}$$

Using those results,the face-vertex correspondence can be written in other forms

$$\tilde{\phi}^{(i)}_{a-\hat{\mu},a}(z_1)R_{12}(z_1-z_2)_{ij}^{i'j'}\phi^{(j')}_{a-\hat{\nu},a}(z_2)$$

$$= \sum_{\kappa} W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\kappa} & a-\hat{\nu} \\ a-\hat{\mu} & a \end{array}\right](z_1-z_2)\phi^{(j)}_{a-\hat{\mu}-\hat{\kappa},a-\hat{\mu}}(z_2)\tilde{\phi}^{(i')}_{a-\hat{\mu}-\hat{\kappa},a-\hat{\nu}}(z_1), \tag{24}$$

$$\bar{\phi}^{(j)}_{a,a+\hat{\mu}}(z_2)R_{12}(z_1-z_2)^{i'j'}_{ij}\phi^{(i')}_{a-\hat{\nu}-\hat{\kappa},a-\hat{\mu}}(z_1)$$
$$= \sum_{\kappa} W\left[\begin{array}{cc} a & a+\hat{\nu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\kappa} \end{array}\right](z_1-z_2)\phi^{(i)}_{a+\hat{\mu},a+\hat{\mu}+\hat{\kappa}}(z_1)\bar{\phi}^{(j')}_{a+\hat{\nu},a+\hat{\mu}+\hat{\kappa}}(z_2), \tag{25}$$

$$\tilde{\phi}^{(i)}_{a-\hat{\mu},a}(z_1)\tilde{\phi}^{(j)}_{a-\hat{\mu}-\hat{\nu},a-\hat{\mu}}(z_2)R_{12}(z_1-z_2)^{i'j'}_{ij}$$
$$= \sum_{\kappa} W\left[\begin{array}{cc} a-\hat{\mu}-\hat{\nu} & a-\hat{\kappa} \\ a-\hat{\mu} & a \end{array}\right](z_1-z_2)\tilde{\phi}^{(j')}_{a-\hat{\kappa},a}(z_2)\tilde{\phi}^{(i')}_{a-\hat{\mu}-\hat{\nu},a-\hat{\kappa}}(z_1), \tag{26}$$

$$\bar{\phi}^{(i)}_{a+\hat{\mu},a+\hat{\mu}+\hat{\nu}}(z_1)\bar{\phi}^{(j)}_{a,a+\hat{\mu}}(z_2)R_{12}(z_1-z_2)^{i'j'}_{ij}$$
$$= \sum_{\kappa} W\left[\begin{array}{cc} a & a+\hat{\kappa} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\nu} \end{array}\right](z_1-z_2)\bar{\phi}^{(j')}_{a+\hat{\kappa},a+\hat{\mu}+\hat{\nu}}(z_2)\bar{\phi}^{(i')}_{a,a+\hat{\kappa}}(z_1). \tag{27}$$

With the help of those face-vertex correspondence relations, we then can construct the $L(z)$ matrix [20-22] which meet the YBR. Let

$$f(a,\mu,z)^j_i = \phi^{(i)}_{a-\hat{\mu},a}(z+\xi_1)\tilde{\phi}^{(j)}_{a-\hat{\mu},a}(z+\xi_2) \tag{28}$$

ξ_1 and ξ_2 are arbitrary complex numbers. Then from the face-vertex correspondence relations Eq.(17) and (26) , we have

$$R(z_1-z_2)^{i'j'}_{ij}f(a-\hat{\mu},\nu,z_1)^{i''}_{i'}f(a,\nu,z_2)^{j''}_{j'}+\mu\longleftrightarrow\nu$$
$$= f(a-\hat{\mu},\nu,z_2)^{j'}_{j}f(a,\nu,z_1)^{i'}_{i}R(z_1-z_2)^{i''j''}_{i'j'}+\mu\longleftrightarrow\nu, \tag{29}$$

$\mu \longleftrightarrow \nu$ means the same form as the term before them, while the μ and ν exchange to each other. We introduce the difference operator Γ_μ

$$\Gamma_\mu f(a) = f(a+\hat{\mu})\Gamma_\mu, \tag{30}$$

and we have

$$\begin{aligned} & R(z_1-z_2)_{ij}^{i'j'}\Gamma_\nu f(a,\nu,z_1)_{i'}^{i''}\Gamma_\mu f(a,\nu,z_2)_{j'}^{j''} + \mu \longleftrightarrow \nu \\ = \ & \Gamma_\nu f(a,\nu,z_2)_j^{j'}\Gamma_\mu f(a,\nu,z_1)_i^{i'} R(z_1-z_2)_{i'j'}^{i''j''} + \mu \longleftrightarrow \nu. \end{aligned} \tag{31}$$

We can find that the above equation is sufficient to prove the following YBR

$$R(z_1-z_2)_{ij}^{i'j'} L(a,z_1)_{i'}^{i''} L(a,z_2)_{j'}^{j''} = L(a,z_2)_j^{j'} L(a,z_1)_i^{i'} R(z_1-z_2)_{i'j'}^{i''j''}. \tag{32}$$

Where $L(a,z_k) = \sum_\mu \Gamma_\mu f(a,\mu,z_k)$. Let

$$g(a,\mu,z)_i^j = \phi_{a,a+\hat{\mu}}^{(i)}(z+\xi_2)\bar{\phi}_{a,a+\hat{\mu}}^{(j)}(z+\xi_1) \tag{33}$$

There is

$$\sum_\nu \Gamma_\nu f(a,\nu,z)_i^j \sum_\mu \Gamma_{-\mu} g(a,\mu,z)_j^k = \delta_{ik} \tag{34}$$

It shows that $L^{-1}(a,z) = \sum_\mu \Gamma_{-\mu} g(a,\mu,z)$. Then the transfer matrix with open boundary conditions is

$$\begin{aligned} t(z) &= tr\tilde{K}(z)L(a,z)K(z)L^{-1}(a,-z) \\ &= tr\tilde{K}(z)\sum_\nu \Gamma_\nu f(a,\nu,z)K(z)\sum_\mu \Gamma_{-\mu} g(a,\mu,-z) \\ &= \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu F_{\mu\nu}^{(1)}(a,z)F_{\mu\nu}^{(2)}(a,z) \\ &= \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu G_{\mu\nu}(a,z), \end{aligned} \tag{35}$$

with

$$F^{(1)}_{\mu\nu}(a,z)=\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z+\xi_2)K(z)^j_i\phi^{(j)}_{a,a+\hat{\mu}}(-z+\xi_2) \tag{36}$$

$$F^{(2)}_{\mu\nu}(a,z)=\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z+\xi_1)\tilde{K}(z)^l_k\phi^{(l)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z+\xi_1) \tag{37}$$

$$G_{\mu\nu}(a,z)=F^{(1)}_{\mu\nu}(a,z)F^{(2)}_{\mu\nu}(a,z). \tag{38}$$

IV The calculation of tansfer matrix t(z)

Substituting the $K(z)$(12) into Eq.(36) , we get

$$F^{(1)}_{\mu\nu}(a,z)=\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z)\sum_{\gamma\in Z_n^2}U_{2\gamma}(z)\omega^{2\gamma_1\gamma_2}[g^{2\gamma_2}h^{2\gamma_1}K_0(0)]^j_i\phi^{(j)}_{a,a+\hat{\mu}}(-z). \tag{39}$$

Here we let $\xi_1=\xi_2=0$. We can prove that

$$g^{\beta-1}h^{\alpha}K_0(0)\phi_{a,a+\hat{\mu}}(z)=(-1)^{\beta-1}e^{2\pi\sqrt{-1}\frac{\alpha}{n}(\frac{\alpha\tau}{2}+z-nw\bar{a}_\mu+\frac{1}{2})}\phi_{a,a+\hat{\mu}}(-z+2nw\bar{a}_\mu-\alpha\tau-\beta), \tag{40}$$

(see Apendix). The above Eq.(39) can be written as

$$\begin{aligned} F^{(1)}_{\mu\nu}(a,z) &= \sum_{\gamma\in Z_n^2}U_{2\gamma}(z)e^{2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}(\frac{2\gamma_1\tau}{2}-z-nw\bar{a}_\mu+\frac{1}{2})} \\ &\times \tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z)\phi^{(i)}_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-2\gamma_2-1) \end{aligned} \tag{41}$$

From Eq.(23) we know $\tilde{\phi}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z)$ can be obtained from the inverse of the matrix $\tilde{M}$ whose elements are

$$\begin{aligned} \tilde{M}_{i\lambda}(z) &= \phi^{(i)}_{a+\hat{\mu}-\hat{\lambda},a+\hat{\mu}}(z) \\ &= \theta^{(i)}(z-nw(\bar{a}_\lambda+\delta_{\mu\lambda}-1)). \end{aligned} \tag{42}$$

So we get

$$\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z)\phi^{(i)}_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-(2\gamma_2+1))=\frac{det\tilde{M}'}{det\tilde{M}}. \tag{43}$$

Substituting the ν column elements of matrix $\tilde{M}$ with the corresponding elements of column vector $\phi_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-(2\gamma_2+1))$ while keeping other matrix elements unchanged, we get the matrix $\tilde{M}'$.

Suppose the elments of a matrix A defined as $A_{ij}=\theta^{(i)}(nz_j)$, one can prove that the determinant of the matrix A have the results [20-22]

$$detA=C(\tau)h(\sum_i z_i-\frac{n-1}{2})\prod_{i<k}h(z_i-z_k). \tag{44}$$

Using this result, we get

$$\begin{aligned}
&\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z)\phi^{(i)}_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-(2\gamma_2+1))\\
=\;&\frac{h(-z+w\delta+w(1-n)+\frac{n-1}{2}-w(\bar{a}_\nu+\delta_{\mu\nu}-1+\bar{a}_\mu)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1))}{h(-z+w\delta+w(1-n)+\frac{n-1}{2})}\\
\times\;&\prod_{j\neq\nu}\frac{h(-w(\bar{a}_j+\delta_{\mu j}-1+\bar{a}_\mu)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1))}{h(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))},
\end{aligned} \tag{45}$$

$(\delta=\sum_i\delta_i)$. With the help of the formula

$$\sigma_0(z+\frac{1}{n}(\alpha\tau+\beta))=e^{-2\pi\sqrt{-1}\frac{\alpha}{n}(\frac{\alpha\tau}{2n}+z+\frac{1}{2}+\frac{\beta}{n})}\sigma_{\alpha,\beta}(z,\tau), \tag{46}$$

finally, we get

$$\begin{aligned}
F^{(1)}_{\mu\nu}(a,z) &= \sum_{\gamma\in Z_n^2}U_{2\gamma}(z)e^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}\\
&\times \frac{\sigma_{2\gamma_1,2\gamma_2+1}(-z+w\delta+w(1-n)+\frac{n-1}{2}-w(\bar{a}_\nu+\delta_{\mu\nu}-1+\bar{a}_\mu))}{\sigma_0(-z+w\delta+w(1-n)+\frac{n-1}{2})}\\
&\times \prod_{j\neq\nu}\frac{\sigma_{2\gamma_1,2\gamma_2+1}(-w(\bar{a}_j+\delta_{\mu j}-1+\bar{a}_\mu))}{\sigma_0(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))}.
\end{aligned} \tag{47}$$

Substituting the $\tilde{K}(z)$ into Eq.(37) , considering the ismorphism relation between $\tilde{K}(z)$ and $K(z)$(11), we get

$$F^{(2)}_{\mu\nu}(a,z)=\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z)\sum_{\gamma\in Z_n^2}U_{2\gamma}(-z-\frac{nw}{2})\omega^{2\gamma_1\gamma_2}[g^{2\gamma_2}h^{2\gamma_1}K_0(0)]^l_k\phi^{(l)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(z) \tag{48}$$

Using Eq(40), we have

$$F^{(2)}_{\mu\nu}(a,z) = \sum_{\gamma\in Z_n^2} U_{2\gamma}(-z-\frac{nw}{2})e^{2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}(\frac{2\gamma_1\tau}{2}+z-nw(\bar{a}_\nu+\delta_{\mu\nu}-1)+\frac{1}{2})}$$
$$\times \bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z)\phi^{(k)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(-z+2nw(\bar{a}_\nu+\delta_{\mu\nu}-1)-2\gamma_1\tau-2\gamma_2-1) \quad (49)$$

From Eq.(23), we also know $\bar{\phi}_{a,a+\hat{\mu}}(-z)$ can be obtained from the inverse of the matrix $\bar{M}$ whose elements are

$$\bar{M}_{i\lambda}(-z) = \phi^{(i)}_{a,a+\hat{\lambda}}(-z)$$
$$= \theta^{(i)}(-z-nw\bar{a}_\lambda). \quad (50)$$

So we get

$$\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z)\phi^{(k)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(-z+2nw(\bar{a}_\nu+\delta_{\mu\nu}-1)-2\gamma_1\tau-2\gamma_2-1) = \frac{det\bar{M}'}{det\bar{M}}. \quad (51)$$

Substituting the μ column elements of matrix $\bar{M}$ with the corresponding elements of column vector $\phi_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(-z+2nw(\bar{a}_\nu+\delta_{\mu\nu}-1)-2\gamma_1\tau-2\gamma_2-1)$, holding other matrix elements unchanged, we get the matrix $\bar{M}'$. Using this result of (44), we get

$$\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z)\phi^{(k)}_{a+\hat{\mu}-\hat{\nu},a+\hat{\mu}}(-z+2nw(\bar{a}_\nu+\delta_{\mu\nu}-1)-2\gamma_1\tau-2\gamma_2-1)$$
$$= \frac{h(z+w\delta+\frac{n-1}{2}-w(\bar{a}_\mu+\delta_{\mu\nu}-1+\bar{a}_\nu)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1))}{h(z+w\delta+\frac{n-1}{2})}$$
$$\times \prod_{j\neq\mu}\frac{h(-w(\bar{a}_j+\delta_{\mu\nu}-1+\bar{a}_\nu)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1))}{h(-w(\bar{a}_j-\bar{a}_\mu))}. \quad (52)$$

Similarly, with the help of Eq.(46), we get

$$F^{(2)}_{\mu\nu}(a,z) = \sum_{\gamma\in Z_n^2} U_{2\gamma}(-z-\frac{nw}{2})e^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}$$

$$
\begin{aligned}
&\times \quad \frac{\sigma_{2\gamma_1,2\gamma_2+1}(z+w\delta+\frac{n-1}{2}-w(\bar{a}_\mu+\delta_{\mu\nu}-1+\bar{a}_\nu))}{\sigma_0(z+w\delta+\frac{n-1}{2})} \\
&\times \quad \prod_{j\neq\mu}\frac{\sigma_{2\gamma_1,2\gamma_2+1}(-w(\bar{a}_j+\delta_{\mu\nu}-1+\bar{a}_\nu))}{\sigma_0(-w(\bar{a}_j-\bar{a}_\nu))}.
\end{aligned} \tag{53}
$$

V Taking trigonometric and scaling limit

When $\tau \to \sqrt{-1}\infty$, considering $\frac{2\gamma_1}{n} \neq m$(m is integer number), we can find that

$$
\theta\left[\begin{array}{c}\frac{1}{2}+\frac{2\gamma_1}{n}\\ \frac{1}{2}+\frac{2\gamma_2+1}{n}\end{array}\right](z,\tau) \to f(\tau)e^{2\pi\sqrt{-1}(\frac{2\gamma_1}{n}+\frac{1}{2}+m)(z+\frac{1}{2}+\frac{2\gamma_2+1}{n})}. \tag{54}
$$

Then,

$$
U_{2\gamma}(z) \to e^{2\pi\sqrt{-1}(\frac{2\gamma_1}{n}+\frac{1}{2}+m)z} \tag{55}
$$

$$
\begin{aligned}
F^{(1)}_{\mu\nu}(a,z) \quad &\to \quad \sum_{\gamma_1}\frac{e^{2\pi\sqrt{-1}(\frac{2\gamma_1}{n}+\frac{1}{2}+m)z}e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}}{\sin\pi(-z+w\delta+w(1-n)+\frac{n-1}{2})} \\
&\times \quad \prod_{j\neq\nu}\frac{1}{\sin\pi(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu})} \\
&\times \quad \sum_{\gamma_2}e^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}f^n(\tau)e^{2\pi\sqrt{-1}(\frac{2\gamma_1}{n}+\frac{1}{2}+m)(-z+n-\frac{1}{2}-nw\bar{a}_\mu+2\gamma_2)} \\
&\rightsquigarrow \quad \sum_{\gamma_2}e^{2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}} = \sum_{\gamma_2}(e^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}})^{\gamma_2} = 0.
\end{aligned} \tag{56}
$$

We can see that from $F^{(2)}_{\mu\nu}(a,z)$, we can obtain the same result. So we only pay our attention to the case that $\frac{2\gamma_1}{n} = m$. The next step is to consider this case.

We know there is an arbitrary parameter c in $U_{2\gamma}(z)$. Let $c = \epsilon\tau + c'$ with $\epsilon < \frac{1}{n}$, we have

$$
\theta\left[\begin{array}{c}\frac{1}{2}+\frac{2\gamma_1}{n}\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{array}\right](z+c,\tau) = e^{-2\pi\sqrt{-1}\epsilon(\frac{2\epsilon\tau}{n}+z+c'+\frac{1}{2}+\frac{2\gamma_2}{n})}\theta\left[\begin{array}{c}\frac{1}{2}+\frac{2\gamma_1}{n}+\epsilon\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{array}\right](z+c',\tau), \tag{57}
$$

then

$$
U_{2\gamma}(z) \quad = \quad e^{-2\pi\sqrt{-1}\epsilon z}\frac{\theta\left[\begin{array}{c}\frac{1}{2}+\frac{2\gamma_1}{n}+\epsilon\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{array}\right](z+c',\tau)}{\theta\left[\begin{array}{c}\frac{1}{2}+\frac{2\gamma_1}{n}+\epsilon\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{array}\right](c',\tau)},
$$

$$\rightarrow \quad e^{-2\pi\sqrt{-1}\epsilon z} e^{2\pi\sqrt{-1}(\frac{1}{2}+\frac{2\gamma_1}{n}+\epsilon+m)z} \tag{58}$$

which have no relation to γ_2 and do not include any dynamics variables like $\bar{a}_\mu$, so we handle it as an irrelevant constant when taking trigonometic limit. The conclusion (56) still hold when $c = \epsilon\tau + c'$. Now, we can get($\frac{2\gamma_1}{n} = m$)

$$\begin{aligned} F^{(1)}_{\mu\nu}(a,z) &= \sum_{\gamma_2} \frac{\sin\pi(-z + w\delta + w(1-n) + \frac{n-1}{2} - w(\bar{a}_\nu + \delta_{\mu\nu} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(-z + w\delta + w(1-n) + \frac{n-1}{2})} \\ &\times \prod_{j\neq\nu} \frac{\sin\pi(-w(\bar{a}_j + \delta_{\mu j} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(-w(\bar{a}_j + \delta_{\mu j} - \bar{a}_\nu - \delta_{\mu\nu}))}, \end{aligned} \tag{59}$$

$$\begin{aligned} F^{(2)}_{\mu\nu}(a,z) &= \sum_{\gamma_2} \frac{\sin\pi(z + w\delta + \frac{n-1}{2} - w(\bar{a}_\nu + \delta_{\mu\nu} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(z + w\delta + \frac{n-1}{2})} \\ &\times \prod_{j\neq\mu} \frac{\sin\pi(-w(\bar{a}_j + \delta_{\mu\nu} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(-w(\bar{a}_j - \bar{a}_\mu))}. \end{aligned} \tag{60}$$

For the n conservations do not depend on the spectrum parameter z, when $z \rightarrow -\sqrt{-1}\infty$, we can get the integrable Hamiltonian of the system. So there are

$$\begin{aligned} F^{(1)}_{\mu\nu}(a,z) &= \sum_{\gamma_2} e^{\pi\sqrt{-1}(-w(\bar{a}_\nu+\delta_{\mu\nu}-1+\bar{a}_\mu)+\frac{2\gamma_2+1}{n})} \\ &\times \prod_{j\neq\nu} \frac{\sin\pi(-w(\bar{a}_j + \delta_{\mu j} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(-w(\bar{a}_j + \delta_{\mu j} - \bar{a}_\nu - \delta_{\mu\nu}))}, \end{aligned} \tag{61}$$

$$\begin{aligned} F^{(2)}_{\mu\nu}(a,z) &= \sum_{\gamma_2} e^{-\pi\sqrt{-1}(-w(\bar{a}_\nu+\delta_{\mu\nu}-1+\bar{a}_\mu)+\frac{2\gamma_2+1}{n})} \\ &\times \prod_{j\neq\mu} \frac{\sin\pi(-w(\bar{a}_j + \delta_{\mu\nu} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})}{\sin\pi(-w(\bar{a}_j - \bar{a}_\mu))}. \end{aligned} \tag{62}$$

Expanding $\sin\pi(-w(\bar{a}_j + \delta_{\mu j} - 1 + \bar{a}_\mu) + \frac{2\gamma_2+1}{n})$ in $F^{(1)}_{\mu\nu}(a,z)$ and $\sin\pi(-w(\bar{a}_j + \delta_{\mu\nu} - 1 + \bar{a}_\nu) + \frac{2\gamma'_2+1}{n})$ in $F^{(2)}_{\mu\nu}(a,z)$ as the follows resepectively,

$$\frac{1}{2\sqrt{-1}}[e^{\pi\sqrt{-1}(-w(\bar{a}_j+\delta_{\mu j}-1+\bar{a}_\mu)+\frac{2\gamma_2+1}{n})} - e^{-\pi\sqrt{-1}(-w(\bar{a}_j+\delta_{\mu j}-1+\bar{a}_\mu)+\frac{2\gamma_2+1}{n})}], \tag{63}$$

$$\frac{1}{2\sqrt{-1}}[e^{\pi\sqrt{-1}(-w(\bar{a}_j+\delta_{\mu\nu}-1+\bar{a}_\nu)+\frac{2\gamma_2'+1}{n})}-e^{-\pi\sqrt{-1}(-w(\bar{a}_j+\delta_{\mu\nu}-1+\bar{a}_\nu)+\frac{2\gamma_2'+1}{n})}], \tag{64}$$

sum over γ_2 and γ_2' and we can find that all the terms of $F^{(1)}_{\mu\nu}(a,z)$ and $F^{(2)}_{\mu\nu}(a,z)$ have the forms

$$\sum_{\gamma_2} e^{\frac{2\pi\sqrt{-1}}{n}m\gamma_2}, m=1\underbrace{\pm 1\pm\cdots\pm 1}_{n-1}=n,n-2,n-4,\cdots,-n+2, \tag{65}$$

and

$$\sum_{\gamma_2'} e^{\frac{2\pi\sqrt{-1}}{n}m\gamma_2'}, m=1\underbrace{\pm 1\pm\cdots\pm 1}_{n-1}=n,n-2,n-4,\cdots,-n+2. \tag{66}$$

Only for case $m=\pm n,0$, the above summations are not equal to zeroes. When $n=$ *even*,m can take n or 0, but it is rather complex to calculate the case $m=0$. For simplicity, we only consider $n=odd$, so m can only take n. We have

$$F^{(1)}_{\mu\nu}(a,z)\rightsquigarrow e^{\pi\sqrt{-1}\sum_i(-w(\bar{a}_i+\delta_{\mu i}-1+\bar{a}_\mu)+\frac{1}{n})}\prod_{j\neq\nu}\frac{1}{\sin pi(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))}, \tag{67}$$

$$F^{(2)}_{\mu\nu}(a,z)\rightsquigarrow e^{-\pi\sqrt{-1}\sum_i(-w(\bar{a}_i+\delta_{\mu\nu}-1+\bar{a}_\nu)+\frac{1}{n})}\prod_{j\neq\mu}\frac{1}{\sin\pi(-w(\bar{a}_j-\bar{a}_\mu))}, \tag{68}$$

$$\begin{aligned} G_{\mu\nu}(a,z) \;&\rightsquigarrow\; e^{\pi\sqrt{-1}(nw(\bar{a}_\nu-\bar{a}_\mu)+nw\delta_{\mu\nu}-w)} \\ &\times \prod_{j\neq\nu}\frac{1}{\sin\pi(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))}\prod_{k\neq\mu}\frac{1}{\sin\pi(-w(\bar{a}_k-\bar{a}_\mu))}. \end{aligned} \tag{69}$$

For $\bar{a}_j-\bar{a}_k=a_j-a_k+\delta_j-\delta_k$, and $H=\sum_{\mu\nu}\Gamma_{-\mu}\Gamma_\nu G_{\mu\nu}(a,z)$, in which $\Gamma_\mu f(a)=f(a+\hat{\mu})\Gamma_\mu$, we get

$$\begin{aligned} H \;&=\; \sum_{\mu\nu}\Gamma_\nu[e^{\pi\sqrt{-1}(nw(a_\nu+\delta_\nu))}\prod_{j\neq\nu}\frac{1}{\sin\pi(-w(a_j-a_\nu+\delta_j-\delta_\nu))}] \\ &\times\; \Gamma_{-\mu}[e^{-\pi\sqrt{-1}(nw(a_\mu+\delta_\mu))}\prod_{k\neq\mu}\frac{1}{\sin\pi(-w(a_k-a_\mu+\delta_k-\delta_\mu))}] \end{aligned} \tag{70}$$

In quantum theory, one can set $\hat{p}_\mu=\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}$,

$$e^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}f(x)=f(x+\frac{\hbar}{\sqrt{-1}}\hat{\mu})e^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}. \tag{71}$$

Compared with $\Gamma_\mu f(a) = f(a+\hat{\mu})\Gamma_\mu$, let $\Gamma_\mu \rightsquigarrow e^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}$, $-\sqrt{-1}w(a_\mu+\delta_\mu) \rightsquigarrow x_\mu$ and $w \rightsquigarrow \hbar$, there is

$$\begin{aligned} H &\rightsquigarrow \sum_{\mu\nu} e^{\hat{p}_\nu}[e^{-n\pi x_\nu}\prod_{j\neq\nu}\frac{1}{\sinh\pi(x_j-x_\nu)}] \\ &\times e^{-\hat{p}_\mu}[e^{n\pi x_\mu}\prod_{k\neq\mu}\frac{1}{\sinh\pi(x_k-x_\mu)}]. \end{aligned} \tag{72}$$

When $\hat{p}_\lambda \rightsquigarrow p_\lambda$, it becomes a Hamiltonan of a classical integrable syetem.

We also can begin directly from Eq.(59) and (60). Acorrding to the expanding method of getting the results (67) and (68), we have

$$\begin{aligned} F^{(1)}_{\mu\nu}(a,z) &\rightsquigarrow [e^{\pi\sqrt{-1}(-z+\frac{n-1}{2}+1-nw\bar{a}_\mu)} - e^{-\pi\sqrt{-1}(-z+\frac{n-1}{2}+1-nw\bar{a}_\mu)}] \\ &\times \prod_{j\neq\nu}\frac{1}{\sin\pi(-w(\bar{a}_j-\bar{a}_\nu+\delta_{\mu j}-\delta\mu\nu))}, \end{aligned} \tag{73}$$

$$\begin{aligned} F^{(2)}_{\mu\nu}(a,z) &\rightsquigarrow [e^{\pi\sqrt{-1}(z+\frac{n-1}{2}+1-nw(\bar{a}_\nu+\delta_{\mu\nu}-1))} - e^{-\pi\sqrt{-1}(z+\frac{n-1}{2}+1-nw(\bar{a}_\nu+\delta_{\mu\nu}-1))}] \\ &\times \prod_{j\neq\mu}\frac{1}{\sin\pi(-w(\bar{a}_j-\bar{a}_\mu))}, \end{aligned} \tag{74}$$

while ignoring those sin functions which have no any dynamics variables. Let

$$g_{\mu\nu} = \prod_{j\neq\nu}\frac{1}{\sin\pi(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))}\prod_{k\neq\mu}\frac{1}{\sin\pi(-w(\bar{a}_k-\bar{a}_\mu))}, \tag{75}$$

we have

$$\begin{aligned} G_{\mu\nu}(a,z) &= \{e^{\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))} + e^{-\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))} \\ &\quad -e^{\pi\sqrt{-1}(-2z-nw(\bar{a}_\mu+1-\bar{a}_\nu-\delta_{\mu\nu}))} - e^{-\pi\sqrt{-1}(-2z-nw(\bar{a}_\mu+1-\bar{a}_\nu-\delta_{\mu\nu}))}\}g_{\mu\nu}. \end{aligned} \tag{76}$$

For different z, $H(z)$ commute with each other, so we can obtain three commutable Hamiltonians H^i from H

$$H^i = \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu G^i_{\mu\nu}, i = 1, 2, 3, \tag{77}$$

with

$$G^1_{\mu\nu} = \{e^{\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))} + e^{-\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))}\}g_{\mu\nu}, \tag{78}$$

$$G^2_{\mu\nu} = e^{\pi\sqrt{-1}(-2z-nw(\bar{a}_\mu+1-\bar{a}_\nu-\delta_{\mu\nu}))}g_{\mu\nu}, \tag{79}$$

$$G^3_{\mu\nu} = e^{-\pi\sqrt{-1}(-2z-nw(\bar{a}_\mu+1-\bar{a}_\nu-\delta_{\mu\nu}))}g_{\mu\nu}. \tag{80}$$

In $G^1_{\mu\nu}$, if we have $\delta_\mu \to \delta_\mu + \rho$, where ρ is cummtable, we may still get two integrable Hamiltonians

$$\begin{aligned} H^{11} &= \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu G^{11}_{\mu\nu}, & (81) \\ H^{12} &= \sum_{\mu\nu} \Gamma_{-\mu}\Gamma_\nu G^{12}_{\mu\nu}, & (82) \end{aligned}$$

with

$$\begin{aligned} G^{11}_{\mu\nu} &= e^{\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))}g_{\mu\nu}, & (83) \\ G^{12}_{\mu\nu} &= e^{-\pi\sqrt{-1}(n+1-nw(\bar{a}_\mu+\bar{a}_\nu+\delta_{\mu\nu}-1))}g_{\mu\nu}. & (84) \end{aligned}$$

H^{11} and H^{12} all can commute with H^2 and H^3, but they may not commute with each other.

An integrable system require its poisson bracket $H = f_1, f_2, \cdots, f_n$ satisfy

$$\{f_i, f_j\} = \sum_{lm}(\frac{\partial f_i}{\partial p_l}\frac{\partial f_j}{\partial q_m} - \frac{\partial f_i}{\partial q_l}\frac{\partial f_j}{\partial p_m}) = 0. \tag{85}$$

When we take variables change, i.e. $q_i \rightarrow q_i' = \chi q_i + \rho_i$, p_i remain unchange, in the new variable set $\{q_i', p_i\}$, the above equation (85) still hold, so the system is still integrable. Taking some kinds of limit after the variable subsitution, we may acheive a new integrable system. i.e. in Eq.(72), let $x_j \rightarrow x_j' = \chi x_j$, there is

$$\begin{aligned} H' &= \sum_{\mu\nu} e^{p_\nu}[e^{-n\pi\frac{1}{\chi}x_\nu'} \prod_{j\neq\nu} \frac{1}{\sinh\frac{\pi}{\chi}(x_j' - x_\nu')}] \\ &\times \ e^{-p_\mu}[e^{n\pi\frac{1}{\chi}x_\mu'} \prod_{k\neq\mu} \frac{1}{\sinh\frac{\pi}{\chi}(x_k' - x_\mu')}]. \end{aligned} \tag{86}$$

When $\chi \rightarrow \infty$, take the lowest order term, we obtain

$$H'' = \sum_{\mu\nu} e^{p_\nu - p_\mu} \prod_{j\neq\nu} \frac{1}{x_j - x_\nu} \prod_{k\neq\mu} \frac{1}{x_k - x_\mu}. \tag{87}$$

This may be an integrable Hamiltonian.

Additionally, we can let $x_0 \rightarrow x_0 + \rho_0$ with others unchanged in H, and let $\sinh(\pi(x_0 - x_i))$ become $\frac{1}{2}e^{\pi(x_0 - x_i)}$ to obtain a new Hamiltonian. We also can change two variables or more to get a new Hamiltonian. i.e. let $\delta_0 \gg \delta_1 \gg \cdots \gg \delta_{n-1} \gg 1 \rightarrow \infty$, $x_i \rightarrow x_i + \delta_i$, there are $\sinh\pi(x_i - x_j) \rightarrow \frac{1}{2}e^{\pi(x_i - x_j + \delta_i - \delta_j))}(i < j)$. Then

$$\begin{aligned} H &\rightarrow \sum_{\mu\nu} e^{p_\nu - p_\mu} e^{-n\pi(x_\mu - x_\nu)} \\ &\times \prod_{j<\nu} e^{-\pi(x_j - x_\nu)} \prod_{j>\nu} e^{-\pi(x_\nu - x_j)} \prod_{k<\mu} e^{-\pi(x_k - x_\nu)} \prod_{k>\mu} e^{-\pi(x_\nu - x_k)}. \end{aligned} \tag{88}$$

Like this, we can get amount of integrable Hamiltonians.

VI Summary and discussions

We study the Z_n Belavin model with the open boundary conditions. By the factorized L operators, we constructed the commuting difference operators which is similar to the

Ruijsenaars-Macdonald's difference operators. We obtained a one-dimensional many-body integrable model from this commuting difference operators. By taking the special limit, this model is similar as the Calogero model.

The one-dimensional integrable models presented in this paper is obtained from the transfer matrix operators with open boundary conditions. Generally, for different solutions of the reflection equations, we can define different transfer matrices. Thus, to some extent, we give a systematic approach to obtain one-dimensional many-body integrable models. Besides the Z_n Belavin model, there are a lot of two-dimensional exactly solvable model in statistical mechanics, it is interesting to find relations between those models with some one-dimensional many-body systems.

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Appendix

$$
\begin{aligned}
& [g^{\beta} h^{\alpha} K_0(0) \phi_{a,a+\hat{\mu}}(z)]^{(i)} \\
= \;& \omega^{\beta i} \delta_{ij} \delta_{j+\alpha,k} \bar{\delta}_{k+l,0} \theta^{(l)}(z - n w \bar{a}_{\mu}, n\tau) \\
= \;& \omega^{\beta i} \theta \begin{bmatrix} \frac{1}{2} + \frac{i+\alpha}{n} \\ \frac{1}{2} \end{bmatrix} (z - n w \bar{a}_{\mu}, n\tau) \\
= \;& \omega^{\beta i} \theta \begin{bmatrix} -\frac{1}{2} - \frac{i+\alpha}{n} \\ -\frac{1}{2} \end{bmatrix} (-z + n w \bar{a}_{\mu}, n\tau) \\
= \;& (-1) \omega^{\beta i} e^{2\pi\sqrt{-1}\frac{i+\alpha}{n}} \theta \begin{bmatrix} \frac{1}{2} - \frac{i+\alpha}{n} \\ \frac{1}{2} \end{bmatrix} (-z + n w \bar{a}_{\mu}, n\tau).
\end{aligned}
$$

Using the formula

$$
\theta^{(i)}(z + \alpha\tau + \beta, n\tau) = e^{-2\pi\sqrt{-1}\frac{\alpha}{2}(\frac{\alpha\tau}{2} + z + \frac{1}{2} + \beta)} e^{2\pi(\frac{1}{2} - \frac{i}{n})\beta} \theta \begin{bmatrix} \frac{1}{2} - \frac{i-\alpha}{n} \\ \frac{1}{2} \end{bmatrix} (z, n\tau) \tag{89}
$$

there is

$$
\begin{aligned}
\cdots \;=\;& (-1)^{(\beta-1)} e^{2\pi\sqrt{-1}\frac{\alpha}{n}(\frac{\alpha\tau}{2} + z - n w \bar{a}_{\mu} + \frac{1}{2})} \omega^{i} \theta \begin{bmatrix} \frac{1}{2} - \frac{i}{n} \\ \frac{1}{2} \end{bmatrix} (-z + n w \bar{a}_{\mu} - \alpha\tau - \beta, n\tau) \\
=\;& (-1)^{(\beta-1)} e^{2\pi\sqrt{-1}\frac{\alpha}{n}\frac{\alpha\tau}{2} + z - n w \bar{a}_{\mu} + \frac{1}{2})} \omega^{i} \phi^{(i)}_{a,a+\hat{\mu}}(-z + 2 n w \bar{a}_{\mu} - \alpha\tau - \beta, n\tau) \\
=\;& (-1)^{(\beta-1)} e^{2\pi\sqrt{-1}\frac{\alpha}{n}(\frac{\alpha\tau}{2} + z - n w \bar{a}_{\mu} + \frac{1}{2})} [g \phi_{a,a+\hat{\mu}}(-z + 2 n w \bar{a}_{\mu} - \alpha\tau - \beta, n\tau)]^{(i)}
\end{aligned} \tag{90}
$$

move the g in the right hand of the equation to the left side and we get the conclusion (40).

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